

EXTENDED SYMMETRIES AND POISSON ALGEBRAS ASSOCIATED TO TWISTED DIRAC STRUCTURES

ALEXANDER CARDONA

To Steven Rosenberg, on his sixtieth birthday.

ABSTRACT. In this paper we study the relationship between the extended symmetries of exact Courant algebroids over a manifold M defined in [1] and the Poisson algebras of admissible functions associated to twisted Dirac structures when acted by Lie groups. We show that the usual homomorphisms of Lie algebras between the algebras of infinitesimal symmetries of the action, vector fields on the manifold and the Poisson algebra of observables, appearing in symplectic geometry, generalize to natural maps of Leibniz algebras induced both by the extended action and compatible moment maps associated to it in the context of twisted Dirac structures.

MSC(2000): 53C57, 53D17.

Keywords: Twisted Dirac structures, Leibniz algebras, Poisson brackets, moment maps.

1. INTRODUCTION

Let $\mathbb{T}M = TM \oplus T^*M$ denote the standard exact Courant algebroid associated to a smooth manifold M , equipped with the natural symmetric pairing

$$\langle X \oplus \alpha, Y \oplus \beta \rangle = \frac{1}{2}(i_X \beta + i_Y \alpha), \quad (1)$$

where $X \oplus \alpha, Y \oplus \beta \in \Gamma(\mathbb{T}M)$, and the twisted Dorfman bracket

$$[X \oplus \alpha, Y \oplus \beta]_H = [X, Y] \oplus (\mathcal{L}_X \beta - i_Y d\alpha - i_Y i_X H), \quad (2)$$

where the twisting is given by the closed 3-form H on M . Let $\mathbb{L}_H < \mathbb{T}M$ be a *Dirac structure* on M , i.e. a sub-bundle of $\mathbb{T}M$ which is involutive under the bracket (2) and maximally isotropic with respect to $\langle \cdot, \cdot \rangle$. The antisymmetrization of the bracket (2) gives rise to the twisted Courant bracket [5][12]

$$[X \oplus \alpha, Y \oplus \beta]_H = [X, Y] \oplus \left(\mathcal{L}_X \beta - \mathcal{L}_Y \alpha - \frac{1}{2}d(i_X \beta - i_Y \alpha) - i_Y i_X H \right), \quad (3)$$

which, when evaluated on sections of \mathbb{L}_H , coincides with the twisted Dorfman bracket (2). Twisted Dirac structures appear naturally in Poisson geometry when, for example, a reduction of a (twisted or non-twisted) Dirac structure is performed [1]. In quantum field theory and superstring theory, the form H has an interpretation as the Neveu-Schwarz 3-form [8]. In [3] it has been shown that, associated to a twisted Dirac structure \mathbb{L}_H , there is a Poisson algebra of *admissible functions* (the case of non-twisted Dirac structures, studied by Courant and Weinstein

Date: April 17, 2012.

in [5][6], is a particular case of this construction). In general, a section $X \oplus \alpha \in \Gamma(\mathbb{L}_H)$ is called an admissible section, or *admissible pair*, if [3]

$$d\alpha + i_X H = 0,$$

and a smooth function f on a manifold M with a twisted Dirac structure \mathbb{L}_H is called *H-admissible* if there exists a smooth vector field X_f on M such that $(X_f, df) \in \Gamma(\mathbb{L}_H)$ is an admissible pair, i.e. if $i_{X_f} H = 0$. We will denote by $C_{\mathbb{L}_H}^\infty(M)$ the Poisson algebra of *H*-admissible functions on M associated to \mathbb{L}_H .

In the case of Dirac structures associated to Poisson and symplectic structures on M , which cannot be twisted, the set of admissible functions is all of $C^\infty(M)$, but in general it is not the case [3]. If a function f is *H*-admissible, we will call a vector field X_f such that (X_f, df) is a section of \mathbb{L}_H a *Hamiltonian vector field* associated to f . In [3] it is shown that, in spite of the fact that Hamiltonian vector fields are not unique in general, the bracket

$$\{f, g\} = \mathcal{L}_{X_f} g \quad (4)$$

defines a Poisson algebra structure on the space $C_{\mathbb{L}_H}^\infty(M)$ of *H*-admissible functions on M (generalizing the classical result of [5]). In this paper we study the relation between the algebra of admissible functions in the twisted case and the notion of moment map associated to extended actions of Lie groups on exact Courant algebroids, defined in [1]. In particular we prove that extended actions on Dirac structures, with compatible moment maps, induce natural equivariant maps on the Lie algebra of vector fields and the Poisson algebra of admissible functions associated to the Dirac structure, giving rise to a relationship between Leibniz algebras and Poisson algebras of functions associated to Dirac structures which generalize the known facts in symplectic and Poisson geometry.

The paper is organized as follows. In section we recall the notion of admissible pair for sections of Courant algebroids and Dirac structures, and the construction of the Poisson algebra associated to a twisted Dirac structure given in [3]. In section 3 we recall the notions of Leibniz and Courant algebras, and we use them to extend some results in [1] to the case of extended actions of Lie groups on exact Courant algebroids twisted by a closed 3-form, together with the notion of moment map associated to such extended actions. In the last section we introduce the notion of Dirac actions and show, in theorem 4.1, under which circumstances the usual morphisms of Lie algebras associated to Hamiltonian actions on symplectic manifolds can be recovered in this context, in terms of morphisms of Leibniz algebras.

2. POISSON ALGEBRAS ASSOCIATED TO TWISTED DIRAC STRUCTURES

Let us consider, for $H \in \Omega^3(M)$ closed, a twisted Dirac structure $\mathbb{L}_H < \mathbb{T}M$ on M , i.e. a sub-bundle of $\mathbb{T}M$ which is involutive under the bracket (2) and maximally isotropic with respect to $\langle \cdot, \cdot \rangle$ [12]. As a first example consider the Dirac structure defined by

$$\mathbb{L}_h = \{(X, i_X h) \in \Gamma(\mathbb{T}M) \mid X \in \mathfrak{X}(M)\}, \quad (5)$$

i.e. the graph in $\mathbb{T}M$ of a non-degenerate 2-form h . It follows from the definition of the twisted bracket (2) that this Dirac structure is integrable if and only if $dh - H = 0$, so that h cannot be closed in general (such a h is called a *H*-closed 2-form in [12]). Particular cases of Dirac manifolds for which $H = 0$ are Poisson and symplectic manifolds (which correspond to graphs, in the generalized tangent bundle $\mathbb{T}M$, of the corresponding Poisson bi-vector and symplectic form, respectively). In these particular cases, the Poisson algebra structure on $C^\infty(M)$ is defined

by the action of Hamiltonian vector fields on smooth functions given by (4). In general, even in the non-twisted case, the Poisson algebra associated to a Dirac structure \mathbb{L} on M can be smaller than $C^\infty(M)$ since (4) defines a Poisson bracket only on admissible functions associated to the Dirac structure, i.e. those functions $f \in C^\infty(M)$ such that $(X_f, df) \in \Gamma(\mathbb{L})$ for some $X_f \in \mathfrak{X}(M)$ (see e.g. [5][6]). A further reduction is necessary in the case of twisted Dirac structures [3].

2.1. Admissible functions in the twisted case. The notion of Dirac manifold (M, \mathbb{L}) as a sub-bundle of $\mathbb{T}M = TM \oplus T^*M$ can be generalized to its higher analogues in $\mathbb{T}^k M = TM \oplus \Lambda^k T^*M$, in the sense of [16], where the integer $k \geq 0$ will be called the order of the Dirac structure. Since $\Gamma(\mathbb{T}^k M) = \mathfrak{X}(M) \oplus \Omega^k(M)$, the twisting in the corresponding Dorfman bracket (2) at order k will be given by a closed $(k+2)$ -form on M . For $X \oplus \alpha, Y \oplus \beta \in \Gamma(\mathbb{T}^k M)$, the twisted Dorfman bracket can be written as

$$[X \oplus \alpha, Y \oplus \beta]_H = \mathcal{L}_X Y + \mathcal{L}_X \beta - i_Y(d\alpha + i_X H), \quad (6)$$

where $H \in \Omega_{cl}^{k+2}(M)$, so that imposing $d\alpha + i_X H = 0$ is equivalent to impose a completely diagonal adjoint action of $X \oplus \alpha$ on $\Gamma(\mathbb{T}^k M)$:

$$\text{ad}_{X \oplus \alpha} = \begin{pmatrix} \mathcal{L}_X & 0 \\ 0 & \mathcal{L}_X \end{pmatrix}. \quad (7)$$

Actually, this is equivalent to consider the couple (X, α) as a geometric symmetry of the differential graded Lie algebra associated to the dg-manifold $\text{Der}^\bullet(T[1]M \oplus \mathbb{R}[k], Q_H)$, where the homological vector field is given by

$$Q_H = d + H\partial_t, \quad (8)$$

d denotes the de Rham differential and $H \in \Omega_{cl}^{k+2}(M)$ is the twisting. As a matter of fact, the twisted Dorfman bracket (2) is known to be the derived bracket obtained from the complex of derivations $\text{Der}^\bullet(T[1]M \oplus \mathbb{R}[k], Q_H)$ (see [11][13][14]). In [3], these facts are used to motivate the following

Definition 2.1. Let \mathbb{L}_H in $\Gamma(\mathbb{T}^k M)$ be a H -twisted Dirac structure of order k , where $H \in \Omega^{k+2}(M)$ is closed. A smooth section $X \oplus \alpha \in \Gamma(\mathbb{L}_H)$ is called an admissible section, or admissible pair, if

$$d\alpha + i_X H = 0. \quad (9)$$

We will denote by $\Gamma_H(\mathbb{T}^k M)$ the space of admissible pairs in $\Gamma(\mathbb{T}^k M)$.

Notice that, when $k = 0$, $\Gamma(\mathbb{T}^0 M) = \mathfrak{X}(M) \oplus C^\infty(M)$ and interpreting the twisting 2-form as a symplectic form on M , condition (9) for a section (X, f) in $\Gamma(\mathbb{T}^0 M)$ is nothing but the usual definition of the Hamiltonian vector field associated to f . When $k = 1$ equality (9) gives rise to a *exact derivation* of the exact Courant algebroid $\mathbb{T}M$, in the sense of [1]. If \mathbb{L}_H is a twisted Dirac structure in $\Gamma(\mathbb{T}^1 M) = \mathfrak{X}(M) \oplus \Omega^1(M)$ it also gives a criterium to identify a Poisson algebra of functions on M [3].

Definition 2.2. A smooth function f on a manifold M with a twisted Dirac structure \mathbb{L}_H is called H -admissible if there exists a smooth vector field X_f on M such that $(X_f, df) \in \Gamma(\mathbb{L}_H)$ is an admissible pair, i.e. if $i_{X_f} H = 0$. We will denote by $C_{\mathbb{L}_H}^\infty(M)$ the space of H -admissible functions on M .

If there is no twisting this definition of admissible function coincides with the one of Courant in [5]. On the other hand, if the twisting is non-trivial, the set of H -admissible functions may be smaller than the space of admissible functions in the usual sense but, as shown in [3], it is a non-trivial Poisson algebra.

Theorem 2.1. *Let f, g be H -admissible functions on M with respect to the twisted Dirac structure \mathbb{L}_H , where $H \in \Omega^3(M)$ is closed. Then the product fg and the bracket $\{f, g\}$ defined by (4) are H -admissible functions. Moreover, such a bracket satisfies both Leibniz and Jacobi identities, and then it defines a Poisson algebra structure on the space $C_{\mathbb{L}_H}^\infty(M)$.*

It is straightforward to see that restricting the twisted Dorfman bracket (2) to admissible pairs (X_f, df) and (X_g, dg) gives [3]

$$[(X_f, df), (X_g, dg)]_H = ([X_f, X_g], d\{f, g\}), \quad (10)$$

generalizing the result already found in [5] in the non-twisted case. Moreover, since

$$i_{[X_f, X_g]}H = \mathcal{L}_{X_f}i_{X_g}H - i_{X_g}\mathcal{L}_{X_f}H = -i_{X_g}di_{X_f}H + i_{X_g}i_{X_f}dH = 0,$$

equation (10) implies that $\{f, g\}$ is H -admissible and

$$[X_f, X_g] = X_{\{f, g\}}. \quad (11)$$

Example 2.1. *Consider the Dirac structure defined in (5), i.e. the graph in $\mathbb{T}M$ of a non-degenerate 2-form h . It follows from the definition of the twisted bracket (2) that this Dirac structure is integrable if and only if $dh - H = 0$, and if the functions $f, g, h \in C_{\mathbb{L}_H}^\infty(M)$ are H -admissible,*

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = H(X_f, X_g, X_h) = 0,$$

so that Jacobi identity holds. Notice that, being a graph of a non-degenerate 2-form, the twisted Dirac structure associates to any function a Hamiltonian vector field X_f , but f is a H -admissible function through the condition $i_{X_f}H = 0$ on X_f . Actually, a pair (X_f, df) in \mathbb{L}_h is H -admissible if and only if $\mathcal{L}_{X_f}h = 0$.

This Poisson algebra of functions is not trivial in general: Let (M, ω) be a symplectic manifold and consider the 2-form $h = \varphi \cdot \omega$, where $\varphi \in C^\infty(M)$ has been chosen to make h non-degenerate. Then the twisted Dirac structure (5) is integrable with respect to the twisted Courant bracket (2) if and only if $H = dh = d\varphi \wedge \omega$. For any $f \in C^\infty(M)$ there exists a vector field $X_f \in \mathfrak{X}(M)$ such that $df = -i_{X_f}h$, $(X_f, -df) \in \Gamma(\mathbb{L}_H)$, but a smooth function f is H -admissible if and only if

$$\mathcal{L}_{X_f}h = \{f, \varphi\}\omega = 0.$$

Thus, in the cases in which φ is the Hamiltonian function for a dynamical system with phase space (M, ω) , an observable $f \in C^\infty(M)$ is H -admissible if and only if it is a constant of motion.

This example suggests that condition (9) is a symmetry condition when applied on functions. In section 4 we will show that this is the case since, actually, it combines both the requirement for an action of a Lie group on M to extend to an action on the exact Courant algebroid $\mathbb{T}M$, and the requirements for this extended action to have a compatible moment map associated to it.

3. EXTENDED ACTIONS AND MOMENT MAPS

Given a smooth action of a Lie group G on a manifold M , let us denote by

$$\psi : \mathfrak{g} \rightarrow \mathfrak{X}(M) \quad (12)$$

the associated infinitesimal action of \mathfrak{g} , the Lie algebra of G , on M , which associates to each element ξ in \mathfrak{g} the corresponding infinitesimal generator of the action $X_\xi \in \mathfrak{X}(M)$. In this section we recall the notion of Leibniz [10] and Courant algebra, and the definition of extension of the infinitesimal action to the exact Courant algebroid $\mathbb{T}M$ [1]. We show that the notion of admissible

pair given in (9) can be used to characterize such extensions and also that \mathfrak{g} -equivariant maps can be used to produce extensions in the case of Courant algebroids twisted by an exact 3-form. The notion of moment map associated to an extended action [1] is also recalled.

3.1. Extended actions of Lie groups. In order to define an extension of the action $\psi : \mathfrak{g} \rightarrow \mathfrak{X}(M)$ to sections of the exact Courant algebroid $\mathbb{T}M = TM \oplus T^*M$, the notion of Courant algebra was introduced in [1]. Recall that a *Leibniz algebra* $(\ell, [\cdot, \cdot]_\ell)$ is an algebra for which the bilinear operation

$$[\cdot, \cdot]_\ell : \ell \times \ell \rightarrow \ell$$

is a derivation, i.e.

$$[a, [b, c]_\ell]_\ell = [[a, b]_\ell, c]_\ell + [b, [a, c]_\ell]_\ell \quad (13)$$

for all $a, b, c \in \ell$ [10]. A morphism of Leibniz algebras is a homomorphism $f : \ell \rightarrow \ell'$ such that

$$f([a, b]_\ell) = [f(a), f(b)]_{\ell'} \quad (14)$$

for all $a, b \in \ell$. A Leibniz algebra $(\ell, [\cdot, \cdot]_\ell)$ for which the bracket $[\cdot, \cdot]_\ell$ is antisymmetric is nothing but a Lie algebra. It follows that, taking the quotient of a Leibniz algebra ℓ by the ideal generated by the brackets of the form $[a, a]_\ell$, for all $a \in \ell$, we obtain a Lie algebra \mathfrak{g}_ℓ , and the quotient map $f_\ell : \ell \rightarrow \mathfrak{g}_\ell$ is a morphism of Leibniz algebras. A natural way to build Leibniz algebras is considering \mathfrak{g} -modules and \mathfrak{g} -equivariant maps, where \mathfrak{g} denotes a Lie algebra. If $\ell_\mathfrak{g}$ is a \mathfrak{g} -module and $\xi \cdot \eta$ denotes the action of $\xi \in \mathfrak{g}$ on $\eta \in \ell_\mathfrak{g}$, an application $\mu : \ell_\mathfrak{g} \rightarrow \mathfrak{g}$ such that

$$\mu(\xi \cdot \eta) = [\mu(\eta), \xi]_\mathfrak{g}$$

induces a Leibniz algebra structure on $\ell_\mathfrak{g}$ given by

$$[\eta, \eta']_{\ell_\mathfrak{g}} = \mu(\eta') \cdot \eta, \quad (15)$$

where $\eta, \eta' \in \ell_\mathfrak{g}$. Here again, the map $\mu : \ell_\mathfrak{g} \rightarrow \mathfrak{g}$ is a morphism of Leibniz algebras. Actually, every Leibniz algebra can be seen as one of this type; this is the model of what is called a Courant algebra in [1] (see [10] for more involved examples and applications of Leibniz algebras).

A *Courant algebra* over a Lie algebra \mathfrak{g} is a Leibniz algebra $(\mathfrak{a}, [\cdot, \cdot]_\mathfrak{a})$ with a morphism $\pi : \mathfrak{a} \rightarrow \mathfrak{g}$ of Leibniz algebras, i.e.

$$\pi([a, b]_\mathfrak{a}) = [\pi(a), \pi(b)]_\mathfrak{g}, \quad (16)$$

for all $a, b \in \mathfrak{a}$. If π is a surjective homomorphism and its kernel $\mathfrak{h} = \ker \pi$ is abelian (with respect to $[\cdot, \cdot]_\mathfrak{a}$) the Courant algebra is called *exact*, and in this case there is a natural \mathfrak{g} -module structure on $\mathfrak{h} = \ker \pi$ given by the adjoint action with respect to $[\cdot, \cdot]_\mathfrak{a}$: If $\eta \in \mathfrak{h}$ and $a \in \mathfrak{a}$ is such that $\pi(a) = \xi$ the map

$$\xi \cdot \eta = [a, \eta]_\mathfrak{a} \quad (17)$$

defines an action of \mathfrak{g} on \mathfrak{h} .

Definition 3.1. *An extension of the action of a Lie group G on a manifold M to the Courant algebroid $\mathbb{T}M$ is an exact Courant algebra \mathfrak{a} over \mathfrak{g} , together with a Courant algebra morphism $\rho : \mathfrak{a} \rightarrow \Gamma(\mathbb{T}M)$ such that \mathfrak{h} acts trivially and the induced action of \mathfrak{g} on $\Gamma(\mathbb{T}M)$ integrates to a G -action on $\mathbb{T}M$.*

Since the Courant algebroid $\mathbb{T}M$ we are working with is exact, as noticed in [1], such an extension gives rise to a commutative diagram

$$\begin{array}{ccccccccc} 0 & \rightarrow & \mathfrak{h} & \rightarrow & \mathfrak{a} & \xrightarrow{\pi} & \mathfrak{g} & \rightarrow & 0 \\ & & \downarrow \nu & & \downarrow \rho & & \downarrow \psi & & \\ 0 & \rightarrow & \Gamma(T^*M) & \rightarrow & \Gamma(\mathbb{T}M) & \xrightarrow{\pi_{TM}} & \Gamma(TM) & \rightarrow & 0 \end{array} \quad (18)$$

in which the image of $\mathfrak{h} = \ker \pi$ under ν is contained in $\Omega_{cl}^1(M)$ and, in order the action to integrate to a G -action over $\mathbb{T}M$, we need a \mathfrak{g} -invariant splitting of $\mathbb{T}M$. It turns out that this condition is equivalent to ask the image of ρ to be given by admissible pairs in the sense of Definition 2.1 (see also [1]):

Proposition 3.1. *Let G be a compact Lie group acting on a smooth manifold M and let $\pi : \mathfrak{a} \rightarrow \mathfrak{g}$ be an exact Courant algebra with a morphism $\rho : \mathfrak{a} \rightarrow \Gamma(\mathbb{T}M)$ such that $\nu(\mathfrak{h}) \subset \Omega_{cl}^1(M)$. Then ρ extends to an action of the Courant algebra \mathfrak{a} if and only if $\rho(\mathfrak{a}) \subset \Gamma_H(\mathbb{T}M)$.*

Proof. Since $\nu(\mathfrak{h}) \subset \Omega_{cl}^1(M)$, it follows from (2) that \mathfrak{h} acts trivially, so we only have to verify that the induced action of \mathfrak{g} on $\mathbb{T}M$ integrates to an action of G . It follows from (7) that, if $\rho(a) \in \Gamma_H(\mathbb{T}M)$ for all $a \in \mathfrak{a}$, the splitting in $\mathbb{T}M = TM \oplus T^*M$ will be preserved and the action will integrate to a G -action on $\mathbb{T}M$. Conversely, given an extended action ρ , the usual averaging argument will give a \mathfrak{g} -invariant splitting for $\mathbb{T}M$ \square

Among the Courant algebras over a Lie algebra \mathfrak{g} induced by \mathfrak{g} -module structures, those induced by semidirect products are particularly useful in order to define extended actions. Consider a \mathfrak{g} -module \mathfrak{h} with left action

$$\cdot : \mathfrak{g} \times \mathfrak{h} \rightarrow \mathfrak{h}.$$

Restricting the adjoint action of \mathfrak{g} on $\mathfrak{g} \ltimes \mathfrak{h}$ we have a Leibniz algebra, the *hemisemidirect product* of \mathfrak{g} with \mathfrak{h} , defined in [9], which will be denoted by $(\mathfrak{a}_{\mathfrak{g}}^{\mathfrak{h}}, \cdot)$, with multiplication given by

$$(\xi, \eta) \cdot (\xi', \eta') = ([\xi, \xi'], \xi \cdot \eta'), \quad (19)$$

for all $(\xi, \eta), (\xi', \eta') \in \mathfrak{g} \oplus \mathfrak{h}$.

Remark 3.1. *Notice that an extended action ρ of the Courant algebra $(\mathfrak{a}, [\cdot, \cdot]_{\mathfrak{a}})$ over \mathfrak{g} on $\mathbb{T}M$ gives rise naturally to a Courant algebra \mathfrak{a}_M over $(\mathfrak{X}(M), [\cdot, \cdot])$, given by $\mathfrak{a}_M = \mathfrak{a}$ and*

$$\Pi_M : \mathfrak{a}_M \rightarrow \mathfrak{X}(M), \quad (20)$$

where $\Pi_M(a) = \pi_{TM}(\rho(a))$ for $a \in \mathfrak{a}$.

Recall that a map $v : \mathfrak{h} \rightarrow \Omega^k(M)$ defined on a \mathfrak{g} -module \mathfrak{h} is called *\mathfrak{g} -equivariant* if

$$v(\xi \cdot \eta) = \mathcal{L}_{X_{\xi}} v(\eta) \quad (21)$$

for all $\xi, \eta \in \mathfrak{a}_{\mathfrak{g}}^{\mathfrak{h}}$. The following proposition shows that equivariant maps give rise to natural extensions of Lie algebra actions to hemisemidirect product algebra actions on *twisted* Courant algebroids.

Proposition 3.2. *Let G be a compact Lie group acting on a smooth manifold M and let \mathfrak{h} be a \mathfrak{g} -module, where \mathfrak{g} denotes the Lie algebra of G . Given a \mathfrak{g} -equivariant map $\mu : \mathfrak{h} \rightarrow C^\infty(M)$, the map $\rho : \mathfrak{a}_{\mathfrak{g}}^{\mathfrak{h}} \rightarrow \Gamma(\mathbb{T}M)$ given by*

$$\rho(\xi, \eta) = (X_{\xi}, \alpha_{(\xi, \eta)}), \quad (22)$$

where $X_{\xi} = \psi(\xi)$ and $\alpha_{(\xi, \eta)} = d\mu(\eta) + i_{X_{\xi}} h$, defines an extended action of the hemisemidirect product (19) on the exact Courant algebroid $\mathbb{T}M$, twisted by an exact 3-form $H = dh$, if and only if $\mathcal{L}_{X_{\xi}} h = 0$ for all $\xi \in \mathfrak{g}$.

Proof. Consider $(\xi, \eta), (\xi', \eta') \in \mathfrak{a}_{\mathfrak{g}}^{\mathfrak{h}}$. Then, using Cartan identities and (19), we find that

$$\begin{aligned} [\rho(\xi, \eta), \rho(\xi', \eta')]_{dh} &= [X_{\xi}, X_{\xi'}] + \mathcal{L}_{X_{\xi}}(d\mu(\eta') + i_{X_{\xi'}} h) - i_{X_{\xi'}} d(d\mu(\eta) + i_{X_{\xi}} h) - i_{X_{\xi}} i_{X_{\xi'}} dh \\ &= X_{[\xi, \xi']} + d\mu(\xi \cdot \eta') + i_{X_{[\xi, \xi']}} h \\ &= \rho((\xi, \eta) \cdot (\xi', \eta')), \end{aligned}$$

since μ is \mathfrak{g} -equivariant. The result follows by Proposition 3.1, since $d\alpha_{(\xi, \eta)} + i_{X_\xi}H = \mathcal{L}_{X_\xi}h$ for all $\xi \in \mathfrak{g}$ \square

Natural examples of extended actions on non-twisted Courant algebroids include the actions commonly used in symplectic geometry. If we consider $\mathfrak{h} = \mathfrak{g}$ in (19), and the adjoint action of \mathfrak{g} on itself, we obtain the exact Courant algebra $\mathfrak{a}_\mathfrak{g}^\mathfrak{g} = \mathfrak{g} \oplus \mathfrak{g}$ over \mathfrak{g} , with bracket

$$[(\xi, \eta), (\xi', \eta')] = ([\xi, \xi'], [\xi, \eta']). \quad (23)$$

Example 3.1. Let M be a smooth manifold and let ω be a closed non-degenerate 2-form on M . Consider the Courant algebroid $\mathbb{T}M$ with $H = 0$, and an infinitesimal action $\psi : \mathfrak{g} \rightarrow \mathfrak{X}(M)$ which integrates to a Lie group action on M . The map $\rho : \mathfrak{a}_\mathfrak{g}^\mathfrak{g} \rightarrow \Gamma(\mathbb{T}M)$ given by

$$\rho(\xi, \eta) = (X_\xi, \alpha_\eta), \quad (24)$$

where $X_\xi = \psi(\xi)$ and $\alpha_\eta = i_{X_\eta}\omega$, will give rise to an extended action whenever $d\alpha_\xi = 0$ for all $\xi \in \mathfrak{g}$. Indeed, as follows from (23),

$$\begin{aligned} [\rho(\xi, \eta), \rho(\xi', \eta')] &= X_{[\xi, \xi']} + i_{X_{[\xi, \eta']}}\omega \\ &= \rho([\xi, \eta], (\xi', \eta')), \end{aligned}$$

for all $\xi, \eta \in \mathfrak{g}$.

3.2. Moment Maps associated to Extended actions. A moment map for an extended \mathfrak{g} -action $\rho : \mathfrak{a} \rightarrow \Gamma(\mathbb{T}M)$ is a \mathfrak{g} -equivariant map

$$\mu : \mathfrak{h} \rightarrow C^\infty(M) \quad (25)$$

such that $\nu = d\mu$, i.e. satisfying

$$d\mu(\xi \cdot \eta) = \mathcal{L}_{\psi(\xi)}d\mu(\eta), \quad (26)$$

where $\psi : \mathfrak{g} \rightarrow \mathfrak{X}(M)$ denotes the infinitesimal action of G over M and $\xi \cdot \eta$ is the action given by (17). In [1] the obstructions to the existence of moment maps associated to extended actions have been studied. It also has been shown that this definition of moment map coincides with the usual one in symplectic geometry when we consider the extended action given by (24) on the Courant algebroid $\mathbb{T}M$, when $H = 0$ and ω is the symplectic form. This definition of moment map is actually equivalent to ask the map $\mu \oplus \psi$ to induce an equivariant map $\rho_o : \mathfrak{a} \rightarrow \Gamma(\mathbb{T}^0M) \cong \mathfrak{X}(M) \oplus C^\infty(M)$ such that the following diagram commutes:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathfrak{h} & \longrightarrow & \mathfrak{a} & \xrightarrow{\pi} & \mathfrak{g} \longrightarrow 0 \\ & & \downarrow \nu & & \downarrow \rho & & \downarrow \psi \\ 0 & \longrightarrow & \Omega^1(M) & \longrightarrow & \Gamma(\mathbb{T}^1M) & \longrightarrow & \mathfrak{X}(M) \longrightarrow 0 \\ & \nearrow \mu & \nearrow \rho_o & \nearrow \psi & & & \\ & 0 & \longrightarrow & C^\infty(M) & \longrightarrow & \Gamma(\mathbb{T}^0M) & \longrightarrow \mathfrak{X}(M) \longrightarrow 0. \end{array} \quad (27)$$

It is interesting to realize that, in this approach, the moment map is no longer attached to the geometry, i.e. to any particular Dirac structure in the exact Courant algebroid $\mathbb{T}M$, but to the extended action itself. As a matter of fact, example 3.1 can be used to show that to

any equivariant map $\mu : \mathfrak{h} \rightarrow C^\infty(M)$, for a \mathfrak{g} -module \mathfrak{h} , it is possible to associate an extended action ρ with moment map μ when $H = 0$ (see proposition 2.17 in [1]). Proposition 3.2 before generalizes such result to the twisted case when the twisting is *exact*. In general, as we will see in section 4, the existence of a moment map associated to an extended action amounts to “reduce” the space $C^\infty(M)$ in (27) to a Poisson algebra of admissible functions with respect to the twisting in $\mathbb{T}M$.

Remark 3.2. In [14] the equivariance of ν appears naturally when a Hamiltonian action of the Lie group G on the dg-manifold $\text{Der}^\bullet(T[1]M \oplus \mathbb{R}[k], Q_H)$ given in (8) is defined as a map of Leibniz algebras

$$\mathfrak{g} \rightarrow G\text{Der}^\bullet(T[1]M \oplus \mathbb{R}[k], Q_H),$$

induced by the map of differential graded Lie algebras associated to the infinitesimal action. Moreover, such maps of Leibniz algebras are characterized in terms of invariant forms in the Cartan model of equivariant cohomology (see [14], lemma 4.8).

4. DIRAC STRUCTURES, ADMISSIBLE FUNCTIONS AND SYMMETRIES

Let $\mathbb{L}_H \leq \mathbb{T}M$ be a H -twisted Dirac structure on M , i.e. a sub-bundle of the exact Courant algebroid $\mathbb{T}M$ which is involutive under the bracket (2) and maximally isotropic with respect to the symmetric pairing (1). Consider an extended action $\rho : \mathfrak{a} \rightarrow \Gamma(\mathbb{T}M)$ associated to an infinitesimal action $\psi : \mathfrak{g} \rightarrow \mathfrak{X}(M)$ of a Lie group G on M .

Definition 4.1. The extended action ρ will be called a Dirac action on \mathbb{L}_H if $\rho(a) \in \Gamma(\mathbb{L}_H)$ for all $a \in \mathfrak{a}$.

Notice that the Dirac structure \mathbb{L}_H will be *preserved* by any Dirac action $\rho : \mathfrak{a} \rightarrow \Gamma(\mathbb{T}M)$ on it, i.e. $[\rho(a), \Gamma(\mathbb{L}_H)]_H \subset \Gamma(\mathbb{L}_H)$ for any a in the Courant algebra \mathfrak{a} . Dirac structures preserved by extended actions give rise to reduced Dirac structures [1]. In this section we will show that, provided the existence of moment maps, Dirac actions induce natural equivariant maps between Courant algebras over the Lie algebra \mathfrak{g} , giving rise to a relationship between Lie algebras and Poisson algebras of functions associated to Dirac structures which generalize the known facts in symplectic and Poisson geometry.

Let us first point out that, if $\rho(a) = (X_a, \alpha_a)$ defines a Dirac action, then the vector field X_a should be a symmetry of the twisting, i.e. $\mathcal{L}_{X_a} H = 0$ for all $a \in \mathfrak{a}$. As consequence of proposition 3.1 and remark 3.1 we have that both tangent and cotangent components of a Dirac action ρ are given by equivariant maps:

Lemma 4.1. Let $\rho(a) = (X_a, \alpha_a)$ denote a Dirac action on a twisted Dirac structure \mathbb{L}_H . Then, for any $a, b \in \mathfrak{a}$,

$$X_{[a,b]_{\mathfrak{a}}} = \mathcal{L}_{X_a} X_b \tag{28}$$

and

$$\alpha_{[a,b]_{\mathfrak{a}}} = \mathcal{L}_{X_a} \alpha_b. \tag{29}$$

Recall that in an exact Courant algebra $\mathfrak{h} = \ker \pi \rightarrow \mathfrak{a} \xrightarrow{\pi} \mathfrak{g}$ the action $\eta \cdot \xi = [a, \xi]_{\mathfrak{a}}$ defines a \mathfrak{g} -module structure on \mathfrak{h} , where $\eta \in \mathfrak{h}$ and $a \in \mathfrak{a}$ is such that $\pi(a) = \xi$. It follows then from (28) and (29) that, in particular, the maps $X : \mathfrak{h} \rightarrow \mathfrak{X}(M)$ and $\alpha : \mathfrak{h} \rightarrow \Omega^1(M)$, defined by each component of the extended action, are equivariant in the sense of (26), i.e. $X_{\xi \cdot \eta} = \mathcal{L}_{X_\xi} X_\eta$ and $\alpha_{\xi \cdot \eta} = \mathcal{L}_{X_\xi} \alpha_\eta$ for all $\eta \in \mathfrak{h}$ and $\xi \in \mathfrak{g}$. Actually (28) was already observed in remark 3.1, and we will show that —provided the existence of a compatible moment map associated to the Dirac

action— equation (29) induces a Courant algebra structure over $(C_{\mathbb{L}_H}^\infty(M), \{\cdot, \cdot\})$, the Poisson algebra given by theorem 2.1.

An admissible function f in the Poisson algebra $C_{\mathbb{L}_H}^\infty(M)$, associated to the H -twisted Dirac structure $\mathbb{L}_H \leq \mathbb{T}M$, is a function for which there exists a vector field X_f such that (X_f, df) is an admissible pair in \mathbb{L}_H , in the sense of definition 2.2. If a Lie group G acts on M by infinitesimal symmetries, and such an action extends to an action $\rho : \mathfrak{a} \rightarrow \Gamma(\mathbb{T}M)$ of a Courant algebra on the exact Courant algebroid $\mathbb{T}M$, we have seen in proposition 3.1 that $\rho(a) = (X_a, \alpha_a)$ is an admissible pair for any $a \in \mathfrak{a}$, i.e. $i_{X_a}H + d\alpha_a = 0$. For example, in the particular case of the extended action $\rho : \mathfrak{g} \oplus \mathfrak{g} \rightarrow \Gamma(\mathbb{T}M)$ given in example 3.1, considering the Dirac structure \mathbb{L}_h associated to a non-degenerate 2-form h (non necessarily closed, see example 2.1) given in (5), with twisting $H = dh$, the condition on ρ to be a Dirac action implies that we have a “diagonal” extended action: $\rho(a) = (X_{\pi(a)}, i_{X_{\pi(a)}}h)$. Thus,

$$\rho([a, a']_{\mathfrak{g} \oplus \mathfrak{g}}) = [\rho(a), \rho(a')] = X_{[\xi, \xi']} + i_{X_{[\xi, \xi']}}h,$$

where $\pi(a) = \xi$ and $\pi(a') = \xi'$. Moreover, since $\rho(a) = X_{\pi(a)} + i_{X_{\pi(a)}}h$ is an admissible pair, it follows that the vector field $X_\xi = \psi(\xi)$ is “locally hamiltonian”, i.e. $\mathcal{L}_{X_\xi}h = 0$ for any $\xi \in \mathfrak{g}$. In this case a moment map $\mu : \mathfrak{g} \rightarrow C^\infty(M)$ for such an extended action will give rise then to *admissible* functions $\mu_\xi \in C_{\mathbb{L}_h}^\infty(M)$. If h is a closed form then for every smooth function f on M there exists a Hamiltonian vector field X_f satisfying $i_{X_f}h + df = 0$, so that $C_{\mathbb{L}_h}^\infty(M) = C^\infty(M)$ and we have the usual morphisms of Lie algebras in symplectic geometry:

$$\begin{array}{ccc} & \mathfrak{g} & \\ \mu \swarrow & & \searrow \mathcal{L} \\ C^\infty(M) & \xrightarrow{\times} & \mathfrak{X}(M) \end{array} \quad (30)$$

associated to the infinitesimal action, where we have used (11).

In general, since a moment map associated to an extended action is defined on the “abelian part” of the Courant algebra $\pi : \mathfrak{a} \rightarrow \mathfrak{g}$, this morphisms occurs very rarely. Given a Dirac action $\rho : \mathfrak{a}_{\mathfrak{g}}^{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{g} \rightarrow \Gamma(\mathbb{L}_H) : a \mapsto (X_a, \alpha_a)$ of the Courant algebra $\mathfrak{a}_{\mathfrak{g}}^{\mathfrak{g}}$ defined in (23) on a twisted Dirac structure \mathbb{L}_H and a moment map $\mu : \mathfrak{g} \rightarrow C^\infty(M)$ for the extended action, we will say that such a moment map is *compatible* with the action whenever

$$\alpha_a = d\mu_{\bar{\pi}(a)}, \quad (31)$$

for all $a \in \mathfrak{a}$, where we denote by $\bar{\pi}(a)$ the pair $(0, \pi(a))$ in $\mathfrak{g} \oplus \mathfrak{g}$ in order to distinguish it from $(0, \pi(a))$, for which $\mu_{\pi(a)} = \mu(\pi(a), 0) = 0$. In this case the morphisms in diagram (30) can be seen as particular cases of the natural Leibniz algebra morphisms (Courant algebras) induced both by the extended action and the moment map compatible with it when $\mathfrak{a} = \mathfrak{a}_{\mathfrak{g}}^{\mathfrak{g}}$.

Theorem 4.1. *Let ρ be an extended action of the Courant algebra $(\mathfrak{a}_{\mathfrak{g}}^{\mathfrak{g}}, [\cdot, \cdot]_{\mathfrak{a}_{\mathfrak{g}}^{\mathfrak{g}}})$ on $\mathbb{T}M$, and let \mathbb{L}_H be a twisted Dirac structure. If ρ is a Dirac action and there exists a moment map $\mu : \mathfrak{g} \rightarrow C^\infty(M)$ compatible with it, then μ induces a Courant algebra structure over $(C_{\mathbb{L}_H}^\infty(M), \{\cdot, \cdot\})$, given by:*

$$\Pi_\mu : \mathfrak{a}_\mu \rightarrow C_{\mathbb{L}_H}^\infty(M), \quad (32)$$

where $\mathfrak{a}_\mu = \mathfrak{a}_{\mathfrak{g}}^{\mathfrak{g}}$ and $\Pi_\mu(a) = \mu_{\bar{\pi}(a)}$ for $a \in \mathfrak{a}_{\mathfrak{g}}^{\mathfrak{g}}$.

Proof. Let $\rho(a) = (X_a, \alpha_a) \in \Gamma(\mathbb{L}_H)$ denote the Dirac action of $\mathfrak{a}_{\mathfrak{g}}^{\mathfrak{g}}$ and let $\mu : \mathfrak{g} \rightarrow C^\infty(M)$ be a moment map compatible with it. Then $\mu(\eta) \in C_{\mathbb{L}_H}^\infty(M)$ for all $\eta \in \mathfrak{g}$ and

$$\begin{aligned} \Pi_\mu([a, b]_{\mathfrak{a}_{\mathfrak{g}}^{\mathfrak{g}}}) &= \mu_{[\pi(a), \pi(b)]} \\ &= \mathcal{L}_{X_{\pi(a)}} \mu_{\pi(b)}, \end{aligned}$$

therefore, by (26) in Lemma 4.1 and the definition (4) of the Poisson bracket in $C_{\mathbb{L}_H}^\infty(M)$,

$$\Pi_\mu([a, b]_{\mathfrak{a}_{\mathfrak{g}}^{\mathfrak{g}}}) = \{\mu_{\pi(a)}, \mu_{\pi(b)}\}, \quad (33)$$

so that Π_μ is a homomorphism of Leibniz algebras and $\mathfrak{a}_{\mathfrak{g}}^{\mathfrak{g}}$ is a Courant algebra on $C_{\mathbb{L}_H}^\infty(M)$ \square

Thus, when the hypothesis of theorem 4.1 are fulfilled, we have a diagram of Leibniz algebra morphisms of the form

$$\begin{array}{ccc} & \mathfrak{a}_{\mathfrak{g}}^{\mathfrak{g}} & \\ & \downarrow \rho & \\ \Pi_\mu \swarrow & \Gamma(\mathbb{T}M) & \searrow \Pi_X \\ & \downarrow \mathcal{L} & \\ C_{\mathbb{L}_H}^\infty(M) & \xrightarrow{\chi} & \mathfrak{X}(M) \end{array} \quad (34)$$

attaching the Lie algebra $\mathfrak{X}(M)$ and the Poisson algebra $C_{\mathbb{L}_H}^\infty(M)$ to the Lie algebra of infinitesimal symmetries \mathfrak{g} . The map ρ_o in the lower row of (27) ensembles the images of the Leibniz algebra maps Π_μ and Π_X as sections of $\mathbb{T}^0 M$. It is clear that diagram (34) becomes (30) when $H = 0$ and the Dirac structure is the graph (5) of a closed non-degenerate 2-form.

Example 4.1. Consider the Dirac structure \mathbb{L}_h defined in (5) as the graph in $\mathbb{T}M$ of the non-degenerate 2-form $h = \varphi \cdot \omega$ where ω denotes a symplectic form on M . This Dirac structure is twisted by $H = d\varphi \wedge \omega$, and $f \in C^\infty(M)$ is admissible if and only if $\mathcal{L}_{X_f} h = \{f, \varphi\}\omega = 0$. Consider an action of a compact Lie group G on M such that φ is invariant, i.e. $\mathcal{L}_{X_\xi} \varphi = 0$ for all $\xi \in \mathfrak{g}$. Then the extended action

$$\rho(a) = (X_{\pi(a)}, i_{\pi(a)} h),$$

for $a \in \mathfrak{a}_{\mathfrak{g}}^{\mathfrak{g}}$ is a Dirac action on \mathbb{L}_h with compatible moment map μ . Since

$$\mathcal{L}_{X_\xi} h = 0$$

for $\xi \in \mathfrak{g}$ it follows that $\mu_\xi \in C_{\mathbb{L}_h}^\infty(M)$ for all $\xi \in \mathfrak{g}$, so that the image of such a moment map is the set of “constants of motion”.

Acknowledgements. The author is grateful to Henrique Bursztyn, Michel Cahen, Simone Gutt, Yoshiaki Maeda and Bernardo Uribe for many stimulating discussions on the geometry of Poisson manifolds and Courant algebroids. This research has been supported by the *Vicerrectoría de Investigaciones* and the *Faculty of Sciences* of the Universidad de los Andes.

REFERENCES

- [1] Bursztyn, H., Cavalcanti, G. and Gualtieri, M. *Reduction of Courant algebroids and generalized complex structures*. Adv. Math., **211**, iss. 2, pp. 726–765, 2007.
- [2] Bursztyn, H. and Weinstein, A. *Poisson geometry and Morita equivalence*. Poisson geometry, deformation quantisation and group representations, pp. 1–78, London Math. Soc. Lecture Note Ser., 323, Cambridge University Press, 2005.
- [3] Cardona, A. *Poisson algebras of admissible functions associated to twisted Dirac structures*. Submitted to *Letters in Mathematical Physics*.
- [4] Cannas da Silva, A. and Weinstein, A. *Geometric models for noncommutative algebras*. Berkeley Mathematics Lecture Notes, **10**. American Mathematical Society, Providence, RI, 1999.
- [5] Courant, T. *Dirac manifolds*. Trans. Amer. Math. Soc. **319**, no. 2, pp. 631–661, 1990.
- [6] Courant, T. and Weinstein, A. *Beyond Poisson structures*. Action hamiltoniennes de groupes. Troisième théorème de Lie (Lyon, 1986), pp. 39–49, Travaux en Cours, **27**, Hermann, Paris, 1988.
- [7] Dorfman I.Y. *Dirac Structures and Integrability of Nonlinear Evolution Equations*. Nonlinear Science: Theory and Applications. Wiley, Chichester, 1993.
- [8] Graña, M. *Flux compactifications and generalized geometries*. Classical Quantum Gravity **23**, no. 21, pp. S883–S926, 2006.
- [9] Kinyon, M. and Weinstein, A. *Leibniz algebras, Courant algebroids, and multiplications on reductive homogeneous spaces*. Amer. J. Math. **123**, no. 3, pp. 525–550, 2001.
- [10] Loday, J-L. *Une version non commutative des algèbres de Lie: les algèbres de Leibniz*. Enseign. Math. **39**, no. 3-4, pp. 269–293, 1993.
- [11] Roytenberg, D. *On the structure of graded symplectic supermanifolds and Courant algebroids*. Contemp. Math. **315**, Amer. Math. Soc., Providence, RI, pp. 169–185, 2002.
- [12] Ševera, P. and Weinstein, A. *Poisson geometry with a 3-form background*. Noncommutative geometry and string theory (Yokohama, 2001). Progr. Theoret. Phys. Suppl. No. 144, pp. 145–154, 2001.
- [13] Ševera, P. *Some title containing the words “homotopy” and “symplectic”, e.g. this one*. Travaux mathématiques. Fasc. XVI, pp.121–137, Univ. Luxemb., Luxembourg, 2005.
- [14] Uribe, B. *Group actions on dg-manifolds and their relation to equivariant cohomology*. Preprint arXiv:1010.5413.
- [15] Vaintrob, A. Yu. *Lie algebroids and homological vector fields*. Russian Math. Surveys **52**, no. 2, pp. 428–429, 1997.
- [16] Zambon, M. *L_∞ -algebras and higher analogues of Dirac structures and Courant algebroids*. Preprint arXiv:1003.1004.

MATHEMATICS DEPARTMENT, UNIVERSIDAD DE LOS ANDES, A.A. 4976 BOGOTÁ, COLOMBIA.
 E-mail address: acardona@uniandes.edu.co